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Kármán–Howarth theorem for the Lagrangian-averaged Navier–Stokes–alpha model of turbulence

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The Lagrangian averaged Navier–Stokes–alpha (LANS$-\alpha$) model of turbulence is found to possess a Kármán–Howarth (KH) theorem for the dynamics of its second-order autocorrelation functions in homogeneous isotropic turbulence. This KH result implies that alpha-filtering in the LANS$-\alpha$ model of turbulence does not affect the exact Navier–Stokes relation between second and third moments at separation distances large compared to the model's length scale α. Moreover, at separations r that are smaller than α, the KH scaling between energy dissipation rate and longitudinal third-order autocorrelation changes to match the scaling found in *two-dimensional* incompressible flow. This is consistent with the corresponding change in scaling of the kinetic energy spectrum from $k^{-5/3}$ for larger scales with $k\alpha < 1$, which switches to k^{-3} for smaller scales with $k\alpha > 1$, as discovered in Foias, Holm & Titi (2001).

1. Introduction

1.1. *Lagrangian-averaged Navier–Stokes–alpha* (*LANS*$-\alpha$) *model*

Recently, a modification of the Navier–Stokes (NS) equations known as the Lagrangian-averaged Navier–Stokes–alpha (LANS$-\alpha$) model was introduced and its steady solutions were compared with experimental data for turbulent flow in pipes and channels in Chen *et al.* (1998, 1999*a*, *b*). These modified equations essentially filter the fluid motion that occurs below a certain length scale (denoted as α), which is a parameter in the model. In the LANS$-\alpha$ model, the length scale alpha is the filter width obtained from inverting the Helmholtz operator $1-\alpha^2\Delta$. Amongst other differences from traditional turbulence modelling approaches, this Helmholtz-filtering approach differs from the large eddy simulation (LES) approach by preserving the basic transport theorems for circulation and vorticity dynamics of the exact NS equations. Comparisons of the LANS$-\alpha$ model of turbulence with the LES approach were investigated for a turbulent mixing layer in Geurts & Holm (2002). Direct numerical simulations (DNS) of the LANS$-\alpha$ model for forced homogeneous turbulence were performed in Chen *et al.* (1999*c*). Decay of homogeneous turbulence was also simulated numerically using this model in Mohseni *et al.* (2000). All these simulations showed the LANS$-\alpha$ model to be considerably less computationally intensive than the exact NS equations, while preserving essentially the same behaviour as NS at length scales larger than α. The basic properties of the LANS$-\alpha$ model and its early development are reviewed in Foias, Holm & Titi (2001). See also Marsden &

Shkoller (2001) and Foias, Holm & Titi (2002) for recent analytical and geometrical results for this model.

1.2. *Kármán–Howarth* (*KH*) *theorem*

The invariant theory of isotropic turbulence was introduced by Kármán & Howarth (1938) and refined by Robertson (1940), who reviewed the Kármán–Howarth (KH) theorem in the light of classical tensor invariant theory. The KH theorem relates the time derivative of the two-point velocity autocorrelation functions to the divergences of the third-order correlation functions. The physical importance of the KH theorem in turbulence modelling is undeniable and the line of investigation that began in 1938 with the KH theorem is still being actively pursued. According to Monin & Yaglom (1975) (vol. II, p. 122) the KH theorem's dynamical equation for the two-point autocorrelation function of the fluid velocity, 'plays a basic part in all subsequent studies in the theory of isotropic turbulence'. A homogeneous (but not necessarily isotropic) version of the KH theorem was discussed by Monin & Yaglom (1975) (vol. II, p. 403), but there was a gap in the proof. Correct proofs were given independently by Frisch (1995) and Lindborg (1996). These proofs were reviewed in Hill (1997), who concentrated on the logical steps needed to eliminate pressure–velocity correlations in the KH theorem without assuming isotropy. See also Hill (2001) and Hill & Boratav (2001).

The KH theorem and its corollary, the $2/15$ law for velocity autocorrelation functions, are proved here for the LANS$-\alpha$ model. These exact results demonstrate how the introduction of the length scale α using Helmholtz filtering affects the dynamics of the velocity autocorrelations as a function of separation between two points fixed in the domain of an isotropic LANS$-\alpha$ flow. The effects turn out to be negligible at separations that are large compared to the filtering scale α ($r \gg \alpha$). In contrast, at separations r smaller than α, we find the scaling between dissipation rate and velocity changes to match that found in two-dimensional incompressible flow. This is consistent with the corresponding change in energy spectrum scaling from $k^{-5/3}$ for $k\alpha < 1$, to k^{-3} for $k\alpha > 1$, that was discovered in Foias *et al.* (2001).

1.3. *The KH theorem and two time scales of turbulence*

Two time scales of interest here may be formed from the mean parameters of turbulence. The first is the cascade time scale, T_1, implicit in Kolmogorov (1941) and given in terms of energy dissipation rate ϵ, wavenumber k and mass M by the rate

$$\frac{1}{T_1} = \left(\frac{\epsilon k^2}{M}\right)^{1/3} \quad \text{(Kolmogorov's cascade rate).} \tag{1.1}$$

Kolmogorov's cascade rate $1/T_1$ increases with wavenumber as $k^{2/3}$ independently of viscosity. So the cascade rate accelerates as it proceeds to smaller length scales. The second time scale is Obukhov's eddy turnover time scale, T_2, given by the rate

$$\frac{1}{T_2} = \left(\frac{E(k)\,k^3}{M}\right)^{1/2} \quad \text{(Obukhov's eddy turnover rate).} \tag{1.2}$$

This estimates the rate at which energy progresses from wavenumber k to wavenumber $2k$. The $E(k) \simeq k^{-5/3}$ scaling of the kinetic energy spectral density emerges from the Kolmogorov (1941) 'theory of locally isotropic turbulence' by using a scaling argument admitted by the $4/5$ law corollary of the Kármán & Howarth (1938) theorem for the structure functions of self-similar, isotropic homogeneous flows of the Navier–

Stokes equations. The $E(k) \simeq k^{-5/3}$ scaling of the kinetic energy spectrum implies that the squared ratio of the two time scales $(T_1/T_2)^2$ is a constant, independent of wavenumber, which Kolmogorov (1941) found to take the value

$$\left(\frac{T_1}{T_2}\right)^2 = C_K \simeq 1.5 \quad \text{(Kolmogorov's constant).} \tag{1.3}$$

The Kármán–Howarth–alpha theorem for the Navier–Stokes–alpha model proved in this paper admits the same $E(k) \simeq k^{-5/3}$ spectral density scaling of the kinetic energy for self-similar flows at $k\alpha < 1$, so that for these larger length scales in the LANS−α model one still has

$$E(k) = \left(\frac{T_1}{T_2}\right)^2 M^{1/3} \epsilon^{2/3} k^{-5/3} = C_K \epsilon^{2/3} k^{-5/3} \quad \text{for} \quad k\alpha < 1. \tag{1.4}$$

However, for $k\alpha > 1$ the Kármán–Howarth–alpha theorem proven here for isotropic homogeneous flows of the LANS−α equations admits, for self-similar flows at the smaller length scales

$$E(k) \simeq k^{-3} \quad \text{for} \quad k\alpha > 1 \text{ in the LANS–}\alpha \text{ model.} \tag{1.5}$$

Consequently, in the LANS−α model, Obukhov's eddy turnover rate $1/T_2$ in (1.2) is constant for small scales satisfying $k\alpha > 1$. Physically, this means length scales that are smaller than α will all be swept along together by the larger scales in the α model. Being swept together by the larger scales, the smaller length scales can phase-lock at the Obukhov eddy turnover time scale near $k\alpha = 1$. Phase-locking allows coherent structures to form at length scales smaller than α. Such coherence is validated here by finding the Hölder index to be unity (Lipschitz continuity) for self-similar LANS−α velocity fields at separations $r < \alpha$. This result is consistent with the estimate of Foias *et al.* (2001),

$$E(k) = \frac{C_K \epsilon^{2/3} k^{-5/3}}{(1 + \alpha^2 k^2)^{2/3}}, \tag{1.6}$$

for the scaling of the LANS−α spectral density for kinetic energy. Estimates of the size of α (smaller than the integral scale and larger than the dissipation scale) are also discussed in Foias *et al.* (2001). The size of α corresponds to the correlation length of Lagrangian fluid trajectories.

If constant $1/T_2$ allows the coherence identified by Lipschitz continuity of the self-similar LANS−α velocity fields at small separations $r < \alpha$, then intermittency and anomalous scaling could be expected to be reduced in the LANS−α model for $k\alpha > 1$. This is consistent with the Fourier transform representation in which the alpha-smoothing retains sweeping of the small scales by the large scales, but reduces the nonlinear interactions amongst the small scales for $k\alpha > 1$. Equation (1.6) follows from an analysis in Foias *et al.* (2001) based on using the symbol $(1 + \alpha^2 k^2)$ of the Helmholtz operator in Fourier space, in combination with an argument of Kraichnan (1967) that associates energy transfer rates and eddy turnover times for the filtered velocity. Physically, introduction of a finite correlation length for Lagrangian fluid trajectories in the α model modifies the nonlinearity to eliminate the faster and faster interactions amongst smaller and smaller scales, while preserving the sweeping of the small eddies by the larger flow structures.

Section 2 reviews the LANS−α model and presents two equivalent formulations of its motion equations that will be instrumental in establishing the KH theorem for its correlation dynamics and its 2/15 law in § 3. Section 4 discusses these results and presents our conclusions and outlook.

2. Two equivalent formulations of the LANS—α model equations

The LANS—α equations are given by

$$\partial_t \boldsymbol{v} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} + v_j \nabla u^j + \nabla(p - \tfrac{1}{2}|\boldsymbol{u}|^2 - \tfrac{1}{2}\alpha^2|\nabla\boldsymbol{u}|^2) = \nu\Delta\boldsymbol{v} + \boldsymbol{f}, \tag{2.1}$$

where $\boldsymbol{v} \equiv \boldsymbol{u} - \alpha^2\Delta\boldsymbol{u}$, $\nabla\cdot\boldsymbol{u} = 0$ and $\boldsymbol{u} = 0$ on the boundary.

In LES terminology, one would call $\boldsymbol{u}$ the filtered velocity and $\boldsymbol{v}$ the 'defiltered' velocity. Both $\boldsymbol{u}$ and $\boldsymbol{v}$ are mean velocities obtained in a closure approximation based on the Taylor hypothesis that turbulent fluctuations are 'frozen' into the mean flow $\boldsymbol{u}$. (Taylor's hypothesis is a vital step in the closure procedure that leads to the NS—α model.) From the viewpoint of the Lagrangian mean theory of Andrews & McIntyre (1978), the difference $\boldsymbol{v} - \boldsymbol{u} = -\alpha^2\Delta\boldsymbol{u}$ is the mean 'pseudomomentum' obtained by using the Taylor hypothesis closure. The divergence-free condition $\nabla\cdot\boldsymbol{u} = 0$ in (2.1) is an additional constraint on $\boldsymbol{u}$. (For constant α, the velocity $\boldsymbol{v}$ is also divergenceless.) Preservation of this constraint will determine the mean pressure p. The unforced, inviscid form of these equations, as well as related equations for geophysical and other flows, first appeared in the context of averaged fluid models in Holm, Marsden & Ratiu (1998*a*, *b*). Their derivation used Lagrangian averaging and asymptotic methods in the variational formulation that modified the pressure terms in equation (2.1). Viscosity was added to the conservative dynamics in Chen *et al.* (1998, 1999*a*, *b*, *c*). Alternative derivations were given in Holm (1999, 2002) and in Marsden & Shkoller (2001). Being a Lagrangian-averaged model, the LANS—α equations (2.1) possess a Kelvin circulation theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\boldsymbol{u})} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = \oint_{c(\boldsymbol{u})} \{\nu\Delta\boldsymbol{v} + \boldsymbol{f}\}\cdot\mathrm{d}\boldsymbol{x}, \tag{2.2}$$

where $c(\boldsymbol{u})$ is a material curve that moves with the filtered velocity.

The LANS—α motion equation (2.1) may be reformulated equivalently as

$$(1 - \alpha^2\Delta)(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla\tilde{p} + \tilde{\boldsymbol{f}} + \alpha^2\,\mathrm{div}\,\tilde{\tau}) = 0 \quad \text{with} \quad \nabla\cdot\boldsymbol{u} = 0. \tag{2.3}$$

Here the stress tensor divergence $\mathrm{div}\,\tilde{\tau}$ is defined by

$$(1 - \alpha^2\Delta)\,\mathrm{div}\,\tilde{\tau} \equiv \mathrm{div}\,\tau \equiv \mathrm{div}(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^T - \nabla\boldsymbol{u}^T\cdot\nabla\boldsymbol{u}). \tag{2.4}$$

Specifying homogeneous boundary conditions in the inversion of the Helmholtz operator $(1 - \alpha^2\Delta)$ in (2.3) allows us to write the LANS—α equation (2.1) both in the interior and on the boundary as

$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla\tilde{p} + \tilde{\boldsymbol{f}} + \alpha^2\,\mathrm{div}\,\tilde{\tau} = 0 \quad \text{with} \quad \nabla\cdot\boldsymbol{u} = 0. \tag{2.5}$$

This has the same form as the NS equations, except for the additional term proportional to α^2. This equation may be rewritten in three-dimensional Cartesian components as

$$\partial_t u_i + \partial_k(u_i u^k + \tilde{p}\delta_i^k) - \nu\Delta u_i = -\alpha^2\partial_k(g_\alpha * \tau_i^k), \quad \text{where} \quad i, j, k \in 1, 2, 3, \tag{2.6}$$

$$\tilde{\tau} = g_\alpha * \tau_i^k \equiv \int g_\alpha(|\boldsymbol{x}' - \boldsymbol{x}|)\,\tau_i^k(\boldsymbol{x}')\,\mathrm{d}^3x', \quad \tau_i^k \equiv (u_{i,j}u^{j,k} + u_{i,j}u^{k,j} - u_{j,i}u^{j,k}). \tag{2.7}$$

Here, g_α is the free-space Green's function for the Helmholtz operator in three dimensions, $g_\alpha(r) = (4\pi\alpha^2 r)^{-1}\exp(-r/\alpha)$, which is the well-known Yukawa potential for $r = |\boldsymbol{x}' - \boldsymbol{x}|$. This free-space Green's function should apply to the extent that the turbulence is isotropic (so that α can be taken as constant) and occurs away from boundaries at distances greater than order $O(\alpha)$.

This free-space form of the LANS−α equation demonstrates that the nonlinear stresses proportional to α^2 are filtered by the Green's function $g_\alpha(r)$ of the Helmholtz operator, whose filter width is $O(\alpha)$. Given α, the term $g_\alpha * \tau_i^k$ is significant at separations r such that $r/\alpha \leqslant 1$ and is negligible for $r/\alpha \gg 1$. For the larger separations, the LANS−α equation reverts to NS. The α-filtering is a regularization. With α-filtering, the solutions of LANS−α are well-posed and have a *finite-dimensional global attractor* under L_2−bounded forcing, as shown in Foias *et al.* (2002) for a periodic domain. See also Marsden & Shkoller (2001) for a proof of their well-posedness in a compact domain. Such well-posedness results are not known to hold for the NS equations. However, solutions of the LANS−α equations do converge to solutions of the NS equations uniformly as $\alpha \to 0$ for any positive viscosity, as was shown in Foias *et al.* (2002) for a periodic domain.

3. The equations governing the LANS−α velocity correlations

In deriving the correlation dynamics, one may regard $(g_\alpha * \tau_j'^{\,k})$ as a subgrid scale (actually, sub-α-scale) stress tensor arising from the alpha-filtering procedure represented by convolution with the free-space Green's function $g_\alpha(r)$, which decreases exponentially in separation with scale length α. See Holm (1999, 2002) and Marsden & Shkoller (2001) for further analysis and explanation of how this viewpoint arises, upon applying Lagrangian averaging. We denote $\boldsymbol{u}(\boldsymbol{x}', t) = \boldsymbol{u}'$ and begin our investigation of the correlation dynamics by computing the ingredients of the partial time derivative $\partial_t(v_i u_j')$,

$$\partial_t v_i + \partial_k(v_i u^k + p\delta_i^k - \alpha^2 u_{i,m} u^{m,k}) = \nu \Delta v_i, \tag{3.1}$$

$$\partial_t u_j' + \partial_k'(u_j' u'^k + \tilde{p}'\delta_j^k) + \alpha^2 \partial_k (g * \tau_j'^{\,k}) = \nu \Delta' u_j'. \tag{3.2}$$

We cross multiply and add these equations, average the result $\overline{(\cdot)}$ and use statistical homogeneity in the following form, with $\xi \equiv \boldsymbol{x}' - \boldsymbol{x}$, in the traditional KH notation:

$$\frac{\partial}{\partial \xi^k}\overline{(\cdot)} = \frac{\partial}{\partial x'^k}\overline{(\cdot)} = -\frac{\partial}{\partial x^k}\overline{(\cdot)}, \tag{3.3}$$

to find the 'Reynolds equation' for the LANS−α model†:

$$\partial_t \overline{(v_i u_j')} - \frac{\partial}{\partial \xi^k}\overline{((v_i u^k - \alpha^2 u_{i,m} u^{m,k}) u_j')} + \frac{\partial}{\partial \xi^k}(\overline{(v_i \tilde{p}')}\delta_j^k - \overline{(u_j' p)}\delta_i^k) + \frac{\partial}{\partial \xi^k}\overline{(v_i(u_j' u'^k + \alpha^2 g * \tau_j'^{\,k}))} = 2\nu \Delta_\xi \overline{(v_i u_j')}, \tag{3.4}$$

where Δ_ξ is the Laplacian operator in the separation coordinate ξ. Next, we symmetrize in i, j and use the relation obtained from homogeneity,

$$\overline{(v_i u_j' u'^k + v_j u_i' u'^k)} = -\overline{(v_i' u_j u^k + v_j' u_i u^k)}, \tag{3.5}$$

† This 'mixed' correlation function is chosen for convenience in interpretation of the results to follow.

to find the homogeneous correlation dynamics for LANS$-\alpha$,

$$\partial_t \overline{(v_i u'_j + v_j u'_i)} - \frac{\partial}{\partial \xi^k}(\mathscr{T}^k_{ij} - \alpha^2 \mathscr{S}^k_{ij} - \Pi^k_{ij}) = 2\nu \Delta_\xi \overline{(v_i u'_j + v_j u'_i)}. \tag{3.6}$$

In this equation, the three symmetric tensors $\mathscr{T}^k_{ij}$, Π^k_{ij} and $\mathscr{S}^k_{ij}$ are defined as

$$\left.\begin{aligned} \Pi^k_{ij} &\equiv \overline{(v_i \tilde{p}')}\,\delta^k_j + \overline{(v_j \tilde{p}')}\,\delta^k_i - \overline{(u'_j p)}\,\delta^k_i - \overline{(u'_i p)}\,\delta^k_j, \\ \mathscr{T}^k_{ij} &\equiv \overline{(v_i u'_j + v_j u'_i + v'_i u_j + v'_j u_i)\, u^k}, \\ \mathscr{S}^k_{ij} &\equiv \overline{(u_{i,m} u'_j + u_{j,m} u'_i) u^{m,k}} + \overline{(v_i g * \tau'^k_j + v_j g * \tau'^k_i)}. \end{aligned}\right\} \tag{3.7}$$

If equation (3.6) is regarded as the Reynolds stress equation for the LANS$-\alpha$ model, then the combination of the three symmetric tensors in (3.7) comprises its stress flux.

3.1. *Imposing isotropy*

We now suppose that the LANS$-\alpha$ solution is isotropic and follow the classical approach of Kármán & Howarth (1938), as refined by Robertson (1940) and Chandrasekhar (1951) using the invariant theory of isotropic tensors. Isotropy implies that we may drop the pressure–velocity tensor Π^k_{ij}. Hence, we rewrite equation (3.6) as

$$\frac{\partial}{\partial t}\mathscr{Q}_{ij} = \frac{\partial}{\partial \xi^k}(\mathscr{T}^k_{ij} - \alpha^2 \mathscr{S}^k_{ij}) + 2\nu \Delta_\xi \mathscr{Q}_{ij}, \tag{3.8}$$

with the corresponding definition, $\mathscr{Q}_{ij} \equiv \overline{(v_i u'_j + v_j u'_i)}$. According to their definitions, both the tensors $\mathscr{Q}_{ij}$ and $\mathscr{T}^k_{ij}$ are symmetric and divergence-free in their indices i, j for constant α. For consistency with the isotropy assumption, equation (3.8) implies that $\mathscr{S}^k_{ij}$ must also be symmetric and divergence-free in its indices i, j.

According to the theory of invariants discussed in Robertson (1940) and Chandrasekhar (1951), these three symmetric, divergence-free, isotropic tensors may each be expressed in terms of a single defining function. In particular, the isotropic u–v autocorrelation tensor $\mathscr{Q}_{ij}$ is given by

$$\mathscr{Q}_{ij} = \mathrm{curl}(Q \varepsilon_{ij\ell} \xi^\ell) = r Q' \left(\frac{\xi_i \xi_j}{r^2} - \delta_{ij}\right) - 2Q\delta_{ij}, \tag{3.9}$$

with defining function $Q(r, t)$ and $Q' = \partial Q/\partial r$ in the KH notation. The isotropic triple correlation tensor is

$$\mathscr{T}^k_{ij} = \mathrm{curl}(T(\xi_i \varepsilon_{jk\ell}\xi^\ell + \xi_j \varepsilon_{ik\ell}\xi^\ell)) = \frac{2}{r} T' \xi_i \xi_j \xi_k - (rT' + 3T)(\xi_i \delta_{jk} + \xi_j \delta_{ik}) + 2T\delta_{ij}\xi_k, \tag{3.10}$$

with defining function $T(r, t)$ and antisymmetric tensor $\varepsilon_{ij\ell}$. Hence, we compute the divergence,

$$\frac{\partial}{\partial \xi^k}\mathscr{T}^k_{ij} = \mathrm{curl}((rT' + 5T)\varepsilon_{ij\ell}\xi^\ell) = \mathrm{curl}\left(\frac{1}{r^4}(r^5 T)' \varepsilon_{ij\ell}\xi^\ell\right). \tag{3.11}$$

This is formula (45) in Chandrasekhar (1951) and is also the corresponding formula (4.13) in Robertson (1940). Likewise, the isotropic mean sub-α-scale stress flux tensor $\mathscr{S}^k_{ij}$ must also take the same form,

$$\frac{\partial}{\partial \xi^k}\mathscr{S}^k_{ij} = \mathrm{curl}((rS' + 5S)\varepsilon_{ij\ell}\xi^\ell) = \mathrm{curl}\left(\frac{1}{r^4}(r^5 S)' \varepsilon_{ij\ell}\xi^\ell\right), \tag{3.12}$$

with defining function $S(r, t)$.

Remark. Due to the presence of curl in their definitions, the defining functions T and $\alpha^2 S$ have dimensions of energy dissipation rate, $(\overline{u^2})^{3/2} r^{-1}$. Note the r^{-1} in T and $\alpha^2 S$ for later comparison with the defining functions in Kármán & Howarth (1938).

3.2. *The equations governing the defining scalars*

According to Robertson (1940), the scalar defining the Laplacian of a second-order isotropic tensor is obtained by operating with

$$\mathrm{D} = \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right) = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \frac{\partial}{\partial r}, \tag{3.13}$$

on the scalar defining the original tensor. That is,

$$\mathrm{D}(Q) = r^{-4}(r^4 Q')'. \tag{3.14}$$

The scalars defining the various second-order tensors in equation (3.8) are, therefore,

$$\frac{\partial Q}{\partial t}, \quad \left(r\frac{\partial}{\partial r} + 5\right) T = \frac{1}{r^4}(r^5 T)', \quad \left(r\frac{\partial}{\partial r} + 5\right) S = \frac{1}{r^4}(r^5 S)', \quad \mathrm{D}(Q). \tag{3.15}$$

As Robertson (1940) points out, upon assuming isotropy, the tensor equation (3.8) is entirely equivalent to the corresponding scalar equation. Substituting the four scalar expressions in (3.15) into equation (3.8) proves the following theorem.

KH THEOREM FOR THE LANS−α MODEL. *Let the LANS−α model flow be homogeneous and isotropic. Then the Reynolds relation* (*3.8*) *is equivalent to the scalar equation,*

$$\frac{\partial Q}{\partial t} = \left(r\frac{\partial}{\partial r} + 5\right)(T - \alpha^2 S) + 2\nu\, \mathrm{D}(Q). \tag{3.16}$$

Remark. Formula (3.16) is the analogue for the LANS−α fluid equations of the KH theorem for NS turbulence.

Upon setting $\partial Q/\partial t = -2\overline{\varepsilon_\alpha}/3$ for the energy dissipation rate in three dimensions and dropping the viscous terms in equation (3.16) – as is appropriate for separation r in the inertial range – one finds the energy balance relation for the LANS−α model,

$$-\tfrac{2}{3}\overline{\varepsilon_\alpha} = \frac{1}{r^4}\frac{\partial}{\partial r}(r^5(T - \alpha^2 S)). \tag{3.17}$$

Here, $\overline{\varepsilon_\alpha}$ denotes the average dissipation rate of the total kinetic energy for the LANS−α model, given by $E_\alpha = \frac{1}{2}\int \boldsymbol{u}\cdot\boldsymbol{v}\, \mathrm{d}^3 x$. Integration of the energy balance relation (3.17) then proves the following corollary.

'2/15 LAW' FOR THE LANS−α MODEL. *In the inertial range and for arbitrary ratio* α/r, *the LANS−α model satisfies the '2/15 law',*

$$-\tfrac{2}{15}\overline{\varepsilon_\alpha} = T - \alpha^2 S. \tag{3.18}$$

Remark. Formula (3.18) is the analogue for the LANS−α fluid equations of Kolmogorov's '4/5 law' for the relation between kinetic energy dissipation rate and longitudinal velocity structure functions in isotropic homogeneous NS turbulence. (Recall that the defining functions T and $\alpha^2 S$ in (3.18) have dimensions of energy dissipation rate, $(\overline{u^2})^{3/2} r^{-1}$, which differ by the factor r^{-1} from the defining functions in Kármán & Howarth (1938) for the NS equations. The factor (2/15)=(1/6)(4/5) arises from the 1/6 relation between autocorrelation functions and structure functions in isotropic turbulence.)

3.3. *Convergence to the NS turbulence theory for $\alpha/r \to 0$*

Foias *et al.* (2002) show that solutions of the LANS$-\alpha$ model converge to solutions of the NS equations as $\alpha \to 0$ in a periodic domain uniformly for any positive viscosity. Therefore, to compare the KH$-\alpha$ theorem with the results of Kármán & Howarth (1938) for the NS equations, we may consider the limit as $\alpha/r \to 0$. In this limit, one may neglect $\alpha^2 S$ in equations (3.17) and (3.18). We follow Robertson (1940) in identifying the KH double correlation scalars $f(r,t)$, $g(r,t)$ as

$$(\overline{u^2})f = Q \quad \text{and} \quad (\overline{u^2})g = \tfrac{1}{2}rQ'. \tag{3.19}$$

Likewise, the KH triple correlation scalars h, k and q are identified in terms of $T(r,t)$, as $(\overline{u^2})^{3/2}h = rT/2$ for h, as well as

$$(\overline{u^2})^{3/2}k = -\frac{1}{r^4}\int_0^r s^4 T \,\mathrm{d}s \quad \text{and} \quad (\overline{u^2})^{3/2}q = \frac{1}{8r^4}\int_0^r s^4 T \,\mathrm{d}s - \frac{r}{4}T, \tag{3.20}$$

for k and q. Using the relation $T = 2(\overline{u^2})^{3/2}h/r$ yields,

$$\left(r\frac{\partial}{\partial r} + 5\right) T = 2(\overline{u^2})^{3/2}\left(\frac{\partial}{\partial r} + \frac{4}{r}\right) h. \tag{3.21}$$

Hence, when $\alpha/r \to 0$, equation (3.16) of the KH$-\alpha$ theorem recovers equation (51) of Kármán & Howarth (1938), namely, the KH equation,

$$(\overline{u^2})\frac{\partial f}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\left[2(\overline{u^2})^{3/2}h + 2\nu(\overline{u^2})\frac{\partial f}{\partial r}\right]. \tag{3.22}$$

One may refer to Monin & Yaglom (1975, p. 122), for the KH equation in their notation. When the factor 1/6 relating third-order autocorrelation functions and structure functions is introduced, this is also equation (3) of Kolmogorov (1941), leading to the Kolmogorov's 4/5 law for Navier–Stokes fluids. See also Landau & Lifschitz (1987) for additional discussion of this fundamental result.

Thus, the limit $\alpha/r \to 0$ of formula (3.16) of the KH$-\alpha$ theorem recovers the expected classical results for homogeneous, isotropic, NS turbulence.

3.4. *Differences from NS turbulence theory for $r < \alpha$*

The second term in the 2/15 law in equation (3.18) (the $\alpha^2\mathscr{S}$ term on the right-hand side) is reminiscent of the quantity that appears in the corresponding '-2 law' for enstrophy cascade in two-dimensional turbulence. The latter expression contains two powers of enstrophy and one power of velocity. For example, see Appendix B of Eyink (1996), where this identity for two-dimensional turbulence is derived in detail. Likewise, the $\alpha^2\mathscr{S}$ term in (3.18) contains two powers of velocity gradient and one of velocity. Consequently, this should be the dominant term (compared to the first $\mathscr{T}$-term) for small separations, when $r < \alpha$. If the LANS$-\alpha$ flow is *self-similar*, the dominance of the $\alpha^2\mathscr{S}$ term in (3.18) when $r < \alpha$ admits the following scaling argument. Following Kolmogorov (1941) as amplified by Frisch (1995), let the longitudinal velocity difference $\delta u_\parallel(\boldsymbol{x},r) = [\boldsymbol{u}(\boldsymbol{x}+\xi) - \boldsymbol{u}(\boldsymbol{x})]\cdot\xi/r$ satisfy the scaling relation $\delta u_\parallel(\boldsymbol{x},\lambda r) = \lambda^h \delta u_\parallel(\boldsymbol{x},r)$ for all $\boldsymbol{x}$ and all increments $r = |\xi|$ and λr small compared to α. By dimensional analysis, $[\mathscr{S}(\lambda r)] = [(\delta u_\parallel)^3/r^3] = [\mathscr{S}(r)]$. Consequently, $3h - 3 = 0$ and, thus, $h = 1$ for small scales $r < \alpha$ in a self-similar LANS$-\alpha$ flow. This means the second-order structure functions follow r^2 scaling for $r < \alpha$ in such a flow. This r^2 scaling implies a k^{-3} law for the kinetic energy spectral density in that range for the LANS$-\alpha$ model. Thus, we find a self-similar k^{-3} 'enstrophy-like' cascade, in agreement with the considerations of Foias *et al.* (2001). The linear law $h = 1$ for its velocity increment at small scales means that the self-similar LANS−α flow velocity is Lipschitz-continuous. This contrasts with Navier–Stokes velocity, for which a similar argument implies Hölder continuity with index $h = 1/3$.

4. Conclusions

The KH theorem for the LANS−α model in equation (3.16) and its corresponding '2/15 law' in equation (3.18) recover the classical results of Kármán & Howarth (1938) and Kolmogorov (1941) in the limit that $\alpha/r \to 0$. These classical results include the $r^{2/3}$ law for the second velocity moments and thus the $k^{-5/3}$ energy spectrum for self-similar homogeneous isotropic NS turbulence at large scales. Hence, α-filtering leaves these velocity autocorrelation statistics for homogeneous isotropic turbulence undisturbed at sufficiently large separations, $\alpha/r \ll 1$. One may also ask how the α-modification affects higher-order correlation functions in the NS−α model. This question addresses the effects of the α-modification on intermittency. Answering it will require numerical investigation of the higher autocorrelation functions, or structure functions for the LANS−α model. This topic will be discussed elsewhere.

The LANS−α results for the KH theorem and its '2/15 law' differ significantly from classical NS turbulence theory for $r < \alpha$. At these small separations, the 2/15 law for the LANS−α model implies an r^2 law instead of the NS $r^{2/3}$ law for its second velocity moments at small scales. The r^2 law for the second moments implies the Hölder index is unity and, thus, the LANS−α self-similar homogeneous isotropic flow velocity is Lipschitz-continuous at small scales. Lipschitz continuity indicates coherent behaviour, so one may expect intermittency and anomalous scaling to be reduced in the LANS−α model for $k\alpha > 1$. This topic will also be discussed elsewhere. The r^2 law corresponds to an 'enstrophy-like' k^{-3} self-similar energy cascade for $k\alpha > 1$, which sustains the considerations of Foias *et al.* (2001) who first discovered the k^{-3} kinetic energy spectrum of the LANS−α model for $k\alpha > 1$.

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REFERENCES

Andrews, D. G. & McIntyre, M. E. 1978 An exact theory of nonlinear waves on a Lagrangian-mean flow. *J. Fluid Mech.* **89**, 609–646.

Chandrasekhar, S. 1951 The invariant theory of isotropic turbulence in magneto-hydrodynamics. *Proc. R. Soc. Lond.* A **204**, 435–449.

Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S. & Wynne, S. 1998 The Camassa–Holm equations as a closure model for turbulent channel and pipe flows. *Phys. Rev. Lett.* **81**, 5338–5341.

Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S. & Wynne, S. 1999*a* The Camassa–Holm equations and turbulence in pipes and channels. *Physica* D **133**, 49–65.

Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S. & Wynne, S. 1999*b* A connection between the Camassa–Holm equations and turbulence in pipes and channels. *Phys. Fluids* **11**, 2343–2353.

Chen, S. Y., Foias, C., Holm, D. D., Margolin, L. G. & Zhang, R. 1999*c* Direct numerical simulations of the Navier–Stokes alpha model. *Physica* D **133**, 66–83.

Eyink, G. 1996 Exact results on stationary turbulence in 2D: consequences of vorticity conservation. *Physica* D **91**, 97–142.

Foias, C., Holm, D. D. & Titi, E. S. 2001 The Navier–Stokes-alpha model of fluid turbulence. *Physica* D **152**, 505–519.

Foias, C., Holm, D. D. & Titi, E. S. 2002 The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. *J. Dyn. Diffl Equat.* **14**, 1–35.

Frisch, U. 1995 *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press.

Geurts, B. J. & Holm, D. D. 2002 Alpha-modeling strategy for LES of turbulent mixing. In *Turbulent Flow Computation* (ed. D. Drikakis & B. G. Geurts). Kluwer.

Hill, R. J. 1997 Applicability of Kolmogorov's and Monin's equations of turbulence. *J. Fluid Mech.* **353**, 67–81.

Hill, R. J. 2001 Equations relating structure functions of all orders. *J. Fluid Mech.* **434**, 379–388.

Hill, R. J. & Boratav, O. N. 2001 Next-order structure functions. *Phys. Fluids* **13**, 276–283.

Holm, D. D. 1999 Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion. *Physica* D **133**, 215–269.

Holm, D. D. 2002 Averaged Lagrangians and the mean dynamical effects of fluctuations in continuum mechanics. *Physica* D (to appear).

Holm, D. D. Marsden, J. E. & Ratiu, T. S. 1998*a* Euler–Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.* **80**, 4173–4177.

Holm, D. D., Marsden, J. E. & Ratiu, T. S. 1998*b* The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. Maths* **137**, 1–81.

von Kármán, T. & Howarth, L. 1938 On the statistical theory of isotropic turbulence. *Proc. R. Soc. Lond.* A **164**, 192–215.

Kolmogorov, A. N. 1941 Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, 16–18 (reprinted in *Proc. R. Soc. Lond.* A **434** (1991), 15–17).

Kraichnan, R. H. 1967 Inertial ranges in two-dimensional turbulence. *Phys. Fluids* **10**, 1417–1423.

Landau, L. D. & Lifschitz, E. M. 1987 *Fluid Mechanics*. 2nd Edn. Pergamon.

Lindborg, E. 1996 A note on Kolmogorov's third-order structure-function law, the local isotropy hypothesis and the pressure-velocity correlation. *J. Fluid Mech.* **326**, 343–356.

Marsden, J. E. & Shkoller, S. 2001 Global well-posedness for the Lagrangian averaged Navier–Stokes (LANS$-\alpha$) equations on bounded domains. *Proc. R. Soc. Lond.* A **359**, 1449–1468.

Mohseni, K., Shkoller, S., Kosović, B., Marsden, J. E., Carati, D., Wray, A. & Rogallo, R. 2000 Numerical simulations of homogeneous turbulence using the Lagrangian averaged Navier–Stokes equations. *Center for Turbulence Research*, *Proc. Summer Program, 2000*, pp. 271–283.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics: Mechanics of Turbulence*. MIT Press.

Robertson, H. P. 1940 The invariant theory of isotropic turbulence. *Proc. Camb. Phil. Soc.* **36**, 209–223.